Scalar formulation of central-force percolation in two dimensions

LETTER TO THE EDITOR

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Abstract. A potential formulation of plane elasticity can be used to study a triangular lattice of elastic springs freely rotating at their endpoints. It provides a scalar formalism for central-force percolation. This emphasises the fact that the specificities of elastic percolation problems lie in fundamental properties such as invariance and symmetry, and not in the vectorial nature of the quantities used in the usual formulations. We map the central-force percolation problem for in-plane deformation onto a flexion problem of super-rigid and elastic plaquettes.

Among the classes of percolation problems [1], central-force percolation [2] (hereafter refered to as CFP) is one of the most intriguing. It is very close to other elastic percolation problems with angular elasticity and, like these, describes the elasticity of randomly depleted materials close to their ruin. However, its geometrical properties are substantially different from usual connectivity percolation, i.e. both the usual scalar percolation problem and the angular elasticity percolation problem [3].

Theories for the rigidity of structures made out of bars date back more than a century (see Maxwell [4] for instance). This very same question is still a field of active mathematical research [5]. In particular, the non-local character of rigidity has raised the important question of the eventual difference of universality class between CFP and usual percolation. Numerical simulations of this problem have not yet settled this point conclusively [2, 3, 6].

The wording 'vector percolation', used in the past to refer either to CFP only or to elastic problems as a whole, has introduced a misleading connection between the vectorial nature of the basic quantities involved—displacements and forces—and the difference between the critical exponents observed for elasticity and conductivity, i.e. a 'scalar' problem. We show in this letter that the vector aspect of the CFP problem has no intrinsic meaning, as it can be expressed as a scalar problem.

The difference between the scalar problem and the CFP problem is still visible when the problems are linearised. This shows that the difference is not caused by some difference in a potential, as the ϕ^4 potential is different from the Gaussian one for magnetic systems. Rather, the difference is accounted for by more fundamental properties such as invariance and symmetries.

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In order to describe the CFP problem, let us consider a triangular lattice of elastic springs free to rotate at their endpoints, i.e. the sites of the lattice. Only a fraction p of the bonds chosen at random are present, the remaining springs (a fraction 1-p) are removed. Let u_i be the displacement vector at site i. We write the Hamiltonian of the lattice as

$$2H = \sum_{i,j} k[(u_i - u_j) \cdot n_{ij}]^2 \tag{1}$$

where the sum runs over all present bonds ij, n_{ij} is a unit vector aligned with the bond ij and k is the elastic constant of a bond. The conjugate variable of the displacement is a force (say f_{ij} in bond ij) which is

$$f_{ij} = k(n_{ij} \otimes n_{ij}) \cdot (u_i - u_j). \tag{2}$$

One of the basic difficulties associated with CFP is apparent in this formula: the operator that relates f to u is a *projection* operator $(n \otimes n)$. This shows that the 'information' transmitted through a bond is only partial and hence the rigidity is a non-local property. This is also responsible for the difficulties encountered in numerical simulations, in particular close to threshold. In the scalar formulation of this problem, however, the projection operator is not present.

In the following, we will only work in the linear approximation of the CFP problem. This consists in letting the vector n_{ij} refer to the *undeformed* lattice, and not the actual direction of the bond when an exterior force is applied to the network.

A possible, though not frequently used, way to handle the elasticity problem is to minimise H over all admissible fields of force. 'Admissible' means that the local equilibrium relations are fulfilled and that the boundary conditions are satisfied. Finding a complete set of admissible stresses is usually a very hard task. In our lattice problem, this turns out to be rather simple to achieve.

Let us consider a 'cartwheel' structure as outlined in figure 1(a). It consists of a hexagon of bonds subjected to a tension force φ and six radial bonds under compression, $-\varphi$. This cartwheel is in equilibrium. We can characterise this state of stress by the single scalar number φ that we label with the index of the site at the centre of the

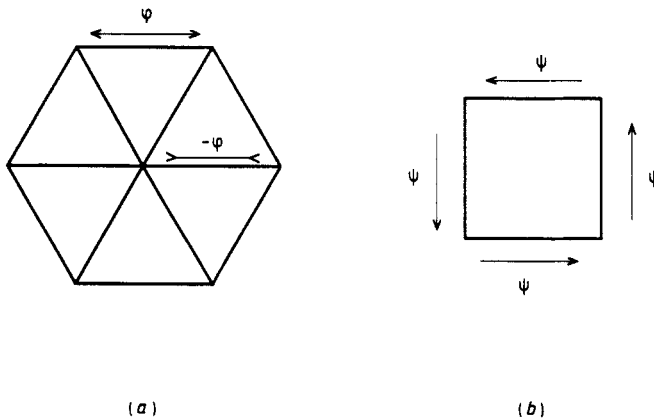(a)   (b)

Figure 1. (a) 'Cartwheel' structure of the prestressed element in the triangular elastic lattice. Each radial bond of the hexagon is compressed with a force $-\varphi$ whereas the perimeter bonds are stretched with the opposite force, φ. (b) The equivalent structure in a conductivity problem is a loop current ψ.

hexagon. Due to linearity, we can superimpose on a lattice any combination of these localised prestressed structures. The resulting force distribution is admissible in the sense that the sum of forces on each site adds up to zero. Now, a simple counting of the degrees of constraints in the lattice gives one per site. If z is the coordination number of the lattice, we have $z/2$ unknown forces and two scalar equations of equilibrium for each node, thus giving one unknown per site for a triangular lattice where $z = 6$. Therefore, all the admissible fields of forces can be generated by the scalar field φ.

Using the variational property is now straightforward: the real intensity of the force in the bond ij, f_{ij}, is simply (see figure 2(a))

$$f_{ij} = \varphi_{k(i,j)} + \varphi_{l(i,j)} - \varphi_i - \varphi_j \tag{3}$$

where $k(i,j)$ and $l(i,j)$ are the sites which form an elementary triangle with i and j, and such that $(i, j, k(i,j))$ and $(j, i, l(i,j))$ are oriented the same way. In order to find the actual distribution of forces in the network, we have to solve

$$H = \frac{1}{2k} \min \sum_{i,j} [\varphi_i + \varphi_j - \varphi_{k(i,j)} - \varphi_{l(i,j)}]^2 \tag{4}$$

where the sum runs over all present bonds and the minimum is taken over the set of φ which satisfies the boundary conditions.
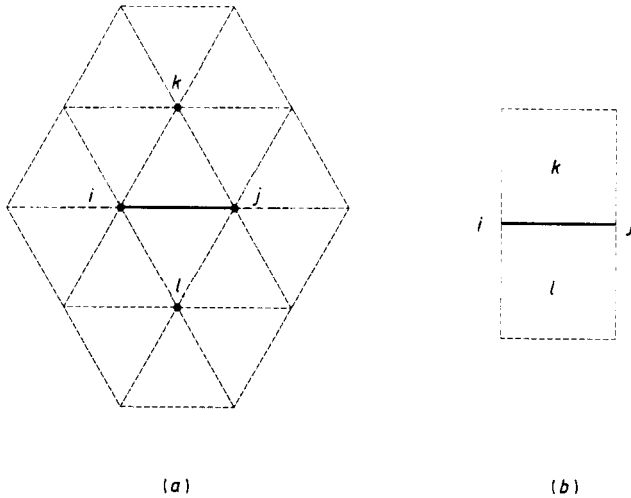

(a) (b)

Figure 2. (a) The real force that is carried by the bond ij is $f_{ij} = \varphi_k + \varphi_l - \varphi_i - \varphi_j$ since ij is radial for the two hexagons centred in i and j on the perimeter of the ones centred at k and l. (b) Similarly, for the conductivity problem, the real current flowing in bond ij is the difference $j_{ij} = (\psi_k - \psi_l)$.

Let us mention here that this approach to the CFP problem is somewhat reminiscent of the potential formulation of plane elasticity through Airy functions [7]. In this formulation the stress tensor σ is related to the Airy potential A by

$$\sigma_{xx} = A_{,yy}$$

$$\sigma_{xy} = -A_{,xy} \tag{5}$$

$$\sigma_{yy} = A_{,xx}$$

and local equilibrium imposes that A is biharmonic, $\nabla^4 A = 0$. The relation between the force f and the potential φ (3) can be seen as the discretisation of a second-order differential operator, as for the Airy potential. More precisely, for a bond oriented along the x axis one can show that f_{ij} will converge, as the lattice spacing a tends to zero, towards

$$f_{ij} \to (a^2/4)(3\varphi_{,yy} - \varphi_{,xx}).$$ (6)

In particular, we can see directly from this equation property that the force distribution is unaffected by the addition of a first-order polynomial in the spatial coordinates. This property is true also for a non-zero lattice spacing and is the potential formulation of the invariance of the elastic Hamiltonian under a rigid motion, i.e. rotation and translation.

The equivalent formalism also works for the random-resistor network. Here, we make use of the loop currents in the network. In figure 1(b), we show for a square lattice such a current distribution ψ, which is the equivalent of the cartwheel structure used in the CFP problem. The Kirchhoff laws are satisfied at every node, and here too, we generate all admissible current distribution in the lattice by a mere superposition of these elementary loop elements that we label by the index of the cell. The real current flowing into a bond ij is the difference between the two loop currents $j_{ij} = (\psi_k - \psi_l)$ of the two adjacent cells shown in figure 2(b). This relation is the equivalent of (3) for the elastic problem.

We notice that, in the random-resistor network, the problem is invariant under the addition of a constant, and not a linear function as in the CFP case, where a first-order polynomial may be added. This is one simple way of viewing the difference between the CFP problem and the conductivity problem.

ψ_k is the loop *current* of cell k in the random-resistor network. The duality transformation, first used in connection with the random-resistor network by Straley [8], maps the cells of the original lattice onto the nodes of a dual lattice. The loop currents of the cells of the original lattice are then interpreted as the *potentials* of the nodes in the dual lattice. Thus, we interpret ψ_k as the potential of the kth node in the dual lattice. Since the currents of the original lattice appear as potentials in the dual lattice, we map the conductivity of the original lattice onto the resistivity of the dual lattice. As a consequence of this, missing bonds in the original lattice map onto superconducting bonds and the usual resistors in the original lattice map onto resistors in the dual lattice. This mapping has proved very useful in the study of the random-resistor network.

With the scalar formulation of the CFP problem given above, a duality transformation corresponding to the one defined in connection with the random-resistor network may be defined. However, in this case the situation is more complex. The duality mapping gives rise to a new scalar problem, whose structure is different from either an elastic problem or a conductivity problem. In the following we give an interpretation of the problem the CFP problem turns into under the duality transformation.

We have seen that the force carried by a bond ij is expressed as a linear combination of the potentials at the four sites ($ijkl$) which form a lozenge whose shortest diagonal is ij. Let us consider a lozenge-shaped plaquette $ijkl$ made out of an elastic material. Furthermore, consider the flexion of this element, obtained by imposing a given displacement z *perpendicular* to the plaquette, to the four vertex sites. In the limit of small displacements z, the flexural elastic energy of the plaquette is

$$E_F = \tfrac{1}{2}h(z_k + z_l - z_i - z_j)^2$$ (7)

since this expression is the only second-order polynomial in z which is invariant under a rigid motion of the plaquette. h is a flexural elastic modulus. Therefore there exists only one mode of deformation for out-of-plane displacements. This mode is shown in figure 3.
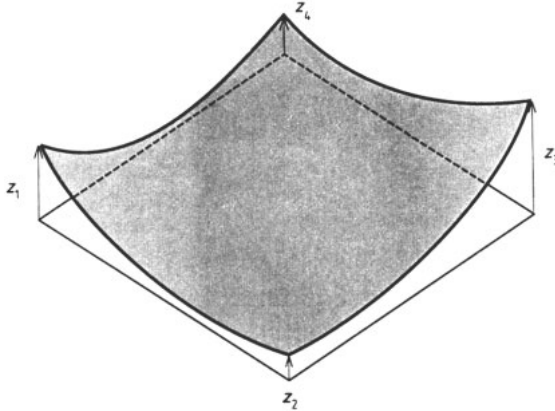

Figure 3. Deformation mode of the plaquettes considered. Only the four corners of the lozenge are connected to the other elements.

This is the problem that the CFP problem maps into under the duality transformation. To each bond of the original triangular lattice, we associate a lozenge-shaped plaquette whose rigidity h is the inverse of the elastic constant k of the bond in the original lattice. Therefore a missing bond in the original lattice will be related to an infinitely rigid plaquette in the dual lattice, in the same way that a superconducting bond in the dual lattice is associated with a missing bond in the original random-resistor network. The corresponding dual lattice of a diluted CFP lattice will have all lozenges occupied, i.e. each triangle will be covered by three plaquettes. The plaquettes are attached together in such a way that all plaquettes sharing a common site i (12 of them) will have the same displacement at the point i, z_i. The potential φ_i at site i is identified with the displacement z_i. For convenience, we list below how the various quantities map into each other with this transformation.

CFP	Flexion
Bond ij	Plaquette $ijkl$
Infinitely rigid	Infinitely soft
Infinitely soft	Infinitely rigid
Potential φ	Displacement z
Force (in-plane)	Angle (out-of-plane)
Elastic energy	Elastic energy
Elastic modulus	Compliance (flexural)

We now mention two consequences of the duality transformation we have just defined. At the central-force percolation threshold, and in-plane deformation, for the random depletion case, the elastic modulus Y goes to zero as [2, 3, 6]

$$Y \propto (p - p^*)^f \tag{8}$$

whereas for the random reinforcement case, i.e. when super-rigid bars are put in with probability p, the rest of the bonds being springs with spring constant k, the elastic modulus diverges as [6, 9]

$$Y \propto (p^* - p)^{-g}. \tag{9}$$

The duality transformation allows us to derive the following critical laws for the flexural modulus F. For the depleted plaquette network with a fraction q of missing elements, F vanishes as

$$F \propto (q^* - q)^g \tag{10}$$

where $q^* = 1 - p^*$. For the super-rigid–elastic plaquette lattice with a fraction q of elastic elements, it diverges as

$$F \propto (q^* - q)^{-f}. \tag{11}$$

Since the exponents f and g may be very different, $f/g \simeq 3$ as obtained in [6], the interchange of the role played by both exponents is remarkable. We note, however, that a similar result is expected also for the case of systems with angular elasticity, e.g. beam lattice, where the critical behaviour occurs at the usual percolation threshold [10].

In addition to making the above duality transformation possible, the scalar formulation may have some other practical applications. For example, it is well known that the dynamics of relaxation processes is critically slowed down close to the threshold, a fact which is particularly true for elasticity problems [11]. This leads to a slow convergence of the numerical methods which relax the 'error' through the lattice. To control the convergence and accelerate it in a very efficient way, elaborate numerical tools have been developed [12]. However, the application of these methods to the case of a vector problem does not provide a significant improvement of raw relaxation methods. A reason that has been proposed to account for this is the decoupling made in the Fourier transform between x and y coordinates [13]. If this is the basic reason, then the scalar formulation presented here should also constitute the basis of an efficient Fourier acceleration relaxation algorithm for this problem.

Another example where this formulation of the CFP problem, or where the plaquette formulation of the dual problem, may be of use is the following. Some models of development of a crack in an elastic medium which mimic DLA processes have been considered by different groups [14]. The real DLA problem is a stochastic growth model in a harmonic potential V (satisfying $\nabla^2 V = 0$) and constant on the cluster already grown. The growth probability is proportional to the potential gradient. The corresponding elastic 'DLA' rupture problem would be a growth phenomenon, in a *bi*harmonic potential (as mentioned for the Airy potential above). Here the growth probability is proportional to the force, i.e. a second differential operator acting on the potential field. The question of the universality class of the resulting structures is still a debated issue. In addition, the results of these studies may, through the present duality transformation, shed light on the corresponding processes in the flexion problem.

We also mention that, for the CFP problem and its dual problem, a field theoretical formulation may be an efficient way of establishing the eventual difference of universality class. This approach may be simplified by the present scalar formulation.

Finally, and more generally, one could study the critical properties of a percolation-type system where transport properties are described by a Hamiltonian H, a function

of a scalar field, and invariant under a given group of transformations $\mathcal{T}$. A general relation between the critical exponent(s) and the set $\mathcal{T}$ would constitute a general framework for transport theory of randomly diluted systems. For elasticity, these transformations are translations and rotations when expressed in terms of the displacement field, and they become the addition of any first-order polynomial to the scalar potential.

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